XVIII. *Obſervations on the Limits of Algebraical Equations; and a general Demonſtration of* Des Cartes's *Rule for finding their Number of affirmative and negative Roots. By the Rev.* Iſaac Milner, *M. A. Fellow of* Queen's College, Cambridge. *Communicated by* Anthony Shepherd, *D. D. F. R. S. and Plumian Profeſſor at* Cambridge.

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§ 1. THE inveſtigations of the limits of equations is conſidered as one of the moſt important problems in algebra. The knowledge of them not only enables us to demonſtrate many uſeful theorems in that ſcience, but is alſo of material ſervice in diſcovering the roots themſelves. Mr. MACLAURIN has treated this ſubject very fully, both in his Algebra and in the Philoſophical Tranſactions.

The ſubſtance of what he has delivered may be briefly expreſſed in the two following propoſitions.

1ſt. That any equation $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$ being propoſed, if you take the fluxion of this equation, and

and divide it by x, the reſulting equation will have all its roots limits of the roots of the given equation.

2dly, If the terms of the propoſed equation be multiplied into the terms of any arithmetical ſeries, the reſulting equation will alſo have its roots limits of the roots of the original equation.

§ 2. This ſecond propoſition, though admitted by all the eminent authors whom I have had an opportunity of conſulting, certainly requires ſome reſtrictions. For example, the roots of the quadratic equation $x^2-2x-3=0$ are 3, &c. -1; multiply the terms of this equation into the terms of the arithmetical progreſſion 1, 2, 3, reſpectively, and the reſulting equation is $1\times x^2-2\times 2x-3\times 3=0$, the roots of which are $2\pm\sqrt{13}$, neither of which are between the roots of the given quadratic.

Again, ſuppoſe the roots of the cubic equation $x^3-px^2+qx-r=0$ to be a, b, $-c$, and it is poſſible that the equation $\overline{l+3m}\times x^3-\overline{l+2m}\times px^2+\overline{l+m}\times qx-lr=0$ may have no root between the quantities b and $-c$; and in general, if the roots of the equation (A) $x^n-px^{n-1}+qx^{n-2}$, &c. $=0$ be ſuppoſed a, b, c, $-d$, $-e$, $-f$, &c. where a is the greateſt root, b the next, and ſo on in order, the equation (B) $\overline{l+nm}\times x^n-\overline{l+n-1.m}px^{n-1}+\overline{l+n-2.m}qx^{n-2}$, &c. $=0$ will not neceſſarily have any of its roots between the roots c and $-d$ of the original equation.

§ 3. It will not be difficult to ſee the reaſon of this, if we examine the demonſtration, which is uſually given us of this ſecond propoſition.

The roots of the biquadratic equation $x^4-Ax^3+Bx^2-Cx+D=0$ are ſuppoſed to be a, b, c, d, and the reſults which ariſe by ſucceſſively ſubſtituting them for x in $4x^3-3Ax^2+2Bx-C$ are ſuppoſed to be $-R$, $+S$, $-T$, $+Z$. From which MACLAURIN concludes, that when $abcd$ are ſubſtituted for x in the quantity $\overline{l+4m}\times x^4-\overline{l+3m}\times Ax^3+\overline{l+2m}Bx^2-\overline{l+m}Cx+lD$, the quantities that reſult will become $-mRx$, $+mSx$, $-mTx$, $+mZx$, where, ſays he, the ſigns being alternately negative and poſitive, it follows, that a, b, c, d, muſt be limits of the equation $\overline{l+4m}\times x^4-\overline{l\times 3m}Ax^3+$ &c. $=0$.

Here it is taken for granted, that the quantities $-mRx$, $+mSx$, $-mTx$, $+mZx$, are alternately negative and poſitive, which is not true, unleſs the roots a, b, c, d, be either all poſitive or all negative.

For ſuppoſe a, b, c, to be poſitive (a) quantities, and d a

(a) Philoſophical Tranſactions, vol. XXXVI. Mr. MACLAURIN, who is here very diffuſe upon this ſubject, never mentions any exception of this ſort.

In his Algebra, art. 44. part 2. he ſays, he ſhall only treat of ſuch equations as have their roots poſitive; but it may be obſerved, that his reaſoning from art. 45. to 50. holds in all equations, the roots of which are real. The theorem in p. 182. of that treatiſe is not general, though applied in the eleventh chapter to the demonſtration of NEWTON's rule for finding impoſſible roots in all equations.

negative

negative one; and then the four refults will be $-m\text{R}a$, $+m\text{S}b$, $-m\text{T}\text{C}$, $-m\text{Z}d$.

§ 4. In general, the roots of the equation $nx^{n-1}-\overline{n-1}\,.\,px^{n-2}+\overline{n-2}\,.\,qx^{n-3}$, are always between the roots of the equation (A) becaufe the roots of this laft equation fubftituted fucceffively for x in $nx^{n-1}-\overline{n-1}\,.\,px^{n-2}$ + &c. always give the refulting quantities alternately negative and pofitive; but when the leaft of the affirmative roots, and the greateft of the negative roots of the equation (A) are fubftituted in (B) the quantities that refult will neceffarily have the fame fign, and therefore it is poffible, that no root of the equation (B) may lie between the leaft of the affirmative and the greateft of the negative roots of the equation (A).

§ 5. It is poffible even, that the equation (B) may have imaginary roots, at the fame time that all the roots of the equation A are real, which is contrary to what all algebraical writers have thought. For inftance, the roots of the equation $x^2+6x-7=0$ are 7 and -1, and if the terms of this equation be multiplied by 1, -1, 3 (an arithmetical feries where the common difference of the terms is equal to 2) the refulting equation will be $x^2-6x+21$, the roots of which are evidently impoffible.

§ 6. However, the equation (B) can never have more than two imaginary roots, when the roots of the equation (A) are real. For fuppofe thefe laft roots to be $+a$, $+b$, $+c$, $+d$, $-e$, $-f$, &c. in their order from the greateft to the leaft, and fince the refults which arife from the fucceffive fubftitution of thefe quantities are always alternately negative and pofitive, that cafe only excepted where d and $-e$ are fubftituted, it is manifeft, that we fhall always have $n-2$ of the roots of the equation (B) which will be limits of the equation (A).

§ 7. It is remarkable, that whenever the equation A has all its terms complete, its roots real, and fome of them pofitive, and others negative, if $l+nm$ be affumed equal to o, the equation B will always have one of its roots either greater than the greateft affirmative root, or lefs than the leaft negative root of the equation (A). Thus, in the quadratic $x^2+6x-7=o$, affume any arithmetical progreffion 0, 1, 2, the firft term of which is equal to nothing, and the equation B in this cafe is $6x-14=o$ and $x=\frac{14}{6}$, which is greater than 1, the greateft affirmative root of the affumed equation.

§ 8. The roots of the equation (A) being ftill fuppofed a, b, c, d, $-e$, $-f$, &c. let m be taken equal to unity, and l any pofitive integer whatfoever, and in that cafe, two of the roots of the equation B will lie between the roots d and $-e$, and one of them will be pofitive, and the other negative.

For

For example, the quadratic equation $x^2+6x-7=0$ has its roots 1 and -7; and if the terms of this equation be multiplied into 3, 2, 1; 4, 3, 2; or 5, 4, 3, ſucceſſively, the reſulting quadratic in every caſe will have its two roots between the roots of the given equation, and one of them will be poſitive, and the other negative.

§ 9. The equation B, which in the laſt article was deduced from the equation A by taking m equal to 1, and l any poſitive integer, may itſelf be treated in the ſame way, and the reſulting equation will, *à fortiori*, have two of its roots between the roots d and $-e$ of the original equation, and one of them will be poſitive, and the other negative.

§ 10. Let $x^2-px+q=0$ repreſent any quadratic equation, the real roots of which are α and β; ſuppoſe $x=\frac{1}{y}$, and we ſhall have $1-py+qy^2=0$, the roots of which equation are $\frac{1}{\alpha}$, $\frac{1}{\beta}$. Let the root of the equation $2qy-p=0$ be equal to $\frac{1}{A}$, and $\frac{1}{A}$ will always lie between the quantities $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and therefore one would think at firſt ſight that the quantity A muſt always lie between α and β. But this would be contrary to what is proved in art. 7. In the preſent caſe A can never lie between α and β, unleſs theſe two quantities have the ſame ſign, and it is obvious,

obvious, that the ſame reaſoning holds in equations of higher dimenſions.

Theſe obſervations, as far as I know, are intirely new. The fundamental propoſition (§ 4.) was, in the year 1775, communicated to Dr. WARING, Lucaſian profeſſor of mathematics in this univerſity, and by him inſerted among the additions to his *Meditationes Algebraicæ* (b).

§ 11. M. EULER, at the concluſion of his 13th chap. *Calcul. Different.* has given a demonſtration of DES CARTES's rule for finding the number of affirmative and of negative roots in any equation, the roots of which are real. From what I have already ſaid, his reaſonings will appear inconcluſive, though I freely own, that what he has done ſuggeſted the following different method.

Suppoſe (D) $L+mx+nx^2+Px^3 \ldots +x^n=0$, and the roots of the equation (E) $m+2nx \ldots\ldots +nx^{n-1}=0$ will be limits of the roots of the equation (D); and therefore there muſt be at leaſt as many poſitive roots in the equation (D) as there are in the equation (E). The ſame may be ſaid of the negative roots: for ſince every root of the equation (E) lies between the different roots of the equation (D), it is impoſſible that the number of roots ſhould be leſs in either caſe. Suppoſe L and Mx to be both poſitive, and ſince the laſt term in

(c) See the end of Proprietates Curv.

any

any equation is always the product of all the roots with their ſigns changed, the number of poſitive roots in each of the equations (D) and (E) muſt be even: therefore, the number of poſitive roots in (D) cannot exceed the number of thoſe in (E) by unity; but there is in (D) one root more than in (E), and conſequently it muſt be negative.

If both the terms L and mx are negative, becauſe then the number of poſitive roots in (E) and (D) are even, it follows in the ſame way, that there is one negative root more in (D) than there is in (E).

And laſtly, if the terms L and mx have different ſigns, for the ſame reaſons there muſt be one poſitive root more in the equation (D) than there is in (E).

DES CARTES' rule is, that there are as many poſitive roots in any equation as there are changes in the ſigns of the terms from + to −, or from − to +, and that the remaining roots are negative. From what has been demonſtrated it appears, that if this rule be true in the equation (E), it muſt hold alſo in the next equation (D) of ſuperior dimenſions; and as we know that it is true in ſimple and quadratic equations, it muſt therefore be true in cubics, in biquadratics, and ſo on.

This is one of the beſt rules we have in algebra. Dr. SAUNDERSON [c] ſaw ſuch an infinity of caſes in equa-

(c) Vol. II. p. 683. Algebra.

tions of high dimenſions, that he ſcarcely hoped for a general proof. MACLAURIN's [d] method is plainly impracticable when the roots are numerous, and therefore this conciſe demonſtration will perhaps be acceptable to mathematicians.

(*d*) Page 145. Algebra.